Basic propositional logic

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In these notes we review the proofs of soundness and completeness for a Hilbert-style proof system for propositional logic. The proof of completeness relies on compactness.

1 Syntax

Definition 1 (Formulas)**.** Let Var be a denumerable set of *propositional variables* $p_1, p_2, p_3, \ldots$. The set Form of formulas is given by the grammar:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A$$

We write $\mathsf{fv}(A)$ for the free variables of A. A context Γ is a set of formulas.

2 Semantics and compactness

Definition 2 (Valuations)**.** A valuation is a function $V : \mathsf{Var} \to \mathbf{2}$. Valuations are extended to formulas $-^V : \mathsf{Var} \to \mathsf{Form}$ as follows:

$$\begin{array}{rcl}
p^V & \stackrel{\text{def}}{=} & V(p) \\
\top^V & \stackrel{\text{def}}{=} & 1 \\
\bot^V & \stackrel{\text{def}}{=} & 0 \\
(\neg A)^V & \stackrel{\text{def}}{=} & 1 - A^V \\
(A \wedge B)^V & \stackrel{\text{def}}{=} & A^V \cdot B^V \\
(A \vee B)^V & \stackrel{\text{def}}{=} & A^V \sqcup B^V \\
(A \to B)^V & \stackrel{\text{def}}{=} & (1 - A^V) \sqcup B^V
\end{array}$$

where $\cdot$ denotes the product and $\sqcup$ denotes the maximum.

Definition 3 (Logical consequence)**.** We write "$\vDash$" for various notions of *logical consequence*:

- $V \vDash A$ holds if $A^V = 1$.
- $V \vDash \Gamma$ holds if $V \vDash A$ for every $A \in \Gamma$.
- $\vDash \Gamma$ holds if $V \vDash \Gamma$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$.
- $\Gamma \vDash A$ holds if $V \vDash \Gamma$ implies $V \vDash A$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$.

We say that Γ is *valid* or a *tautology* if $V \vDash \Gamma$ for every valuation $V : \mathsf{Form} \to \mathbf{2}$. We say that Γ is *satisfiable* if $V \vDash \Gamma$ for some valuation $V : \mathsf{Form} \to \mathbf{2}$, and *unsatisfiable* otherwise. Two formulas A, B are *logically equivalent* if for every valuation $V : \mathsf{Form} \to \mathbf{2}$ $V \vDash A$ holds if and only if $V \vDash B$ holds. Logical equivalence is defined similarly for contexts.

Remark 4. $\Gamma \vDash A$ holds if and only if $\Gamma, \neg A$ is unsatisfiable.

Theorem 5 (Compactness). *The following are equivalent:*

1. Γ *is satisfiable.*
2. Γ' *is satisfiable for every finite subset* $\Gamma' \subseteq \Gamma$

Proof. ($1 \implies 2$) Immediate. ($2 \implies 1$) For each $n \in \mathbb{N}$, let Γ_n be the set of formulas in Γ that use at most the first n propositional variables, *i.e.*

$$\Gamma_n \stackrel{\text{def}}{=} \{A \in \Gamma \mid \mathsf{fv}(A) \subseteq \{p_1, \dots, p_n\}\}$$

The set Γ_n may be infinite, but up to logical equivalence it contains at most 2^{2^n} formulas. So for each n there is a finite set $\Gamma'_n \subseteq \Gamma_n$ logically equivalent to Γ_n. Since Γ'_n is satisfiable, there is a valuation V_n such that $V_n \vDash \Gamma_n$ for all $n \in \mathbb{N}$. Note moreover that if $i \geq n$ then $V_i \vDash \Gamma_n$.

Now for each $n \in \mathbb{N}_0$ we inductively define a set $I_n \subseteq \mathbb{N}$ in such a way that

- I_n is infinite;
- if $i, j \in I_n$ then V_i and V_j coincide on the first n propositional variables; and
- $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$.

To do so, set $I_0 \stackrel{\text{def}}{=} \mathbb{N}$ and

$$I_{n+1} \stackrel{\text{def}}{=} \begin{cases} E_n^1 & \text{if } E_n^1 \text{ is infinite} \\ E_n^0 & \text{otherwise} \end{cases}$$

where $E_n^b = \{i \in I_n \mid i \geq n+1,\ V_i(p_{n+1}) = b\}$.

Now we define a valuation $V : \mathsf{Var} \to \mathbf{2}$ as follows:

$$V(p_n) \stackrel{\text{def}}{=} V_i(p_n) \qquad \text{for any } i \in I_n$$

noting that if $i \geq n$ then $V_i(p_n) = V(p_n)$. To conclude, we claim that $V \vDash \Gamma$. Indeed, let $A \in \Gamma$. Then $A \in \Gamma_n$ for some $n \in \mathbb{N}$. In particular, take $i \in I_n$. Then $V_i \vDash \Gamma_n$ because $i \geq n$, so also $V_i \vDash A$. Moreover $A^{V_i} = A^V$ because A uses only the first n variables, so $V \vDash A$ as required. □

Corollary 6. *If* $\Gamma \vDash A$ *then there is a finite subset* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vDash A$.

Proof. By the previous remark, $\Gamma \vDash A$ holds if and only if the set $\Gamma, \neg A$ is unsatisfiable. So if $\Gamma \vDash A$ by Compactness there exists a finite unsatisfiable subset $\Gamma_0 \subseteq \Gamma, \neg A$. Without loss of generality we may assume that $\neg A \in \Gamma_0$ so $\Gamma_0 = (\Gamma', \neg A)$ is unsatisfiable. This in turn means that $\Gamma' \vDash A$. □

3 Hilbert-style proof system

Definition 7 (Propositional proof system)**.** The only deduction rule is *modus ponens*:

$$\frac{\vdash P : A \to B \qquad \vdash Q : A}{\vdash (P \cdot Q) : B}$$

Plus the following axioms, instantiated on arbitrary formulas A, B, …:

Implication

$$\begin{array}{rcl} K & : & A \to (B \to A) \\ S & : & (A \to (B \to C)) \to (A \to B) \to (A \to C) \end{array}$$

Truth and falsity

$$\begin{array}{rcl} \mathsf{trivial} & : & \top \\ \mathsf{abort} & : & \bot \to A \end{array}$$

Negation

$$\begin{array}{rcll} \mathsf{negi} & : & (A \to B) \to ((A \to \neg B) \to \neg A) & \\ \mathsf{nege} & : & A \to (\neg A \to B) & \textbf{NB. } \text{Redundant if } \mathsf{dneg} \text{ is allowed.} \\ \mathsf{dneg} & : & \neg\neg A \to A & \textbf{NB. } \text{Classical.} \end{array}$$

Conjunction

$$\begin{array}{rcl} \mathsf{pair} & : & A \to (B \to (A \wedge B)) \\ \pi_1 & : & (A \wedge B) \to A \\ \pi_2 & : & (A \wedge B) \to B \end{array}$$

Disjunction

$$\begin{array}{rcl} \mathsf{in}_1 & : & A \to (A \vee B) \\ \mathsf{in}_2 & : & B \to (A \vee B) \\ \mathsf{match} & : & (A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \end{array}$$

We write $\Gamma \vdash A$ if A can be proved using *modus ponens*, the axioms, and hypotheses from Γ. We assume that $\to$ is right-associative and $\cdot$ is left-associative. Application $P \cdot Q$ is written $P\,Q$.

Lemma 8 (Identity)**.** *$\vdash A \to A$ holds for any formula A.*

Proof. It suffices to take $\mathsf{id} \stackrel{\text{def}}{=} S\ K\ K$:

$$\dfrac{\dfrac{\dfrac{}{\vdash S : (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A} \qquad \dfrac{}{\vdash K : A \to (A \to A) \to A}}{\vdash SK : (A \to A \to A) \to A \to A} \qquad \dfrac{}{\vdash K : A \to A \to A}}{\vdash SKK : A \to A}$$

$\square$

Lemma 9 (Weakening)**.** *If* $\Gamma \vdash A$ *and* $\Gamma \subseteq \Gamma'$ *then* $\Gamma' \vdash A$.

Proof. Straightforward by induction on the derivation of $\Gamma \vdash A$. □

Theorem 10 (Deduction Theorem)**.** $\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \to B$.

Proof. The ($\Leftarrow$) direction is immediate since by Weakening (Lem. 9) we have that $\Gamma, A \vdash A \to B$ so:

$$\frac{\Gamma, A \vdash A \to B \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B}$$

For the ($\Rightarrow$) direction we proceed by induction on the derivation of $\Gamma, A \vdash B$. There are three cases, either (1) we use an axiom or an assumption from Γ, (2) we use A, or (3) we use the *modus ponens* rule:

1. **Axiom or fixed assumption.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{}{\Gamma, A \vdash B}$$

   where B is one of the axioms (K, S, trivial, etc.) or $B \in \Gamma$. Then B can also be proved under Γ so we have:

$$\frac{\Gamma \vdash K : B \to A \to B \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

2. **Identity.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{}{\Gamma, A \vdash A}$$

   Then by the Identity lemma (Lem. 8) we have:

$$\frac{}{\Gamma \vdash A \to A}$$

3. **Modus ponens.** That is, the derivation of $\Gamma, A \vdash B$ is of the form:

$$\frac{\Gamma, A \vdash C \to B \qquad \Gamma, A \vdash C}{\Gamma, A \vdash B}$$

   So by *i.h.* on each of the premises we have:

$$\frac{\dfrac{\dfrac{}{\Gamma \vdash S : (A \to (C \to B)) \to (A \to C) \to (A \to B)} \qquad \dfrac{i.h.}{\Gamma \vdash A \to (C \to B)}}{\Gamma \vdash (A \to C) \to (A \to B)} \qquad \dfrac{i.h.}{\Gamma \vdash A \to C}}{\Gamma \vdash A \to B}$$

□

3.1 Basic facts

Following, we state and prove some basic principles.

1. **Contrapositive ($\Gamma, A \to B \vdash \neg B \to \neg A$).**
   By the Deduction Theorem (Thm. 10), it suffices to show that $\Gamma, x : A \to B, y : \neg B \vdash \neg A$. Indeed:

$$\Gamma, x : A \to B, y : \neg B \vdash \mathsf{negi}\, x\, (\underbrace{K y}_{A \to \neg B}) : \neg A$$

2. **Explosion principle ($\Gamma, A, \neg A \vdash B$).** This is an immediate consequence of the Deduction Theorem (Thm. 10) and axiom nege. Moreover, if dneg is allowed, nege is redundant:

$$\Gamma, x : A, y : \neg A \vdash \mathsf{dneg}\, \underbrace{(\mathsf{negi}\, (\underbrace{K x}_{\neg B \to A})\, (\underbrace{K y}_{\neg B \to \neg A}))}_{\neg\neg B} : B$$

3. **False antecedent ($\Gamma, \neg A \vdash A \to B$).** Immediate from the explosion principle and the Deduction Theorem (Thm. 10).

4. **True consequent ($\Gamma, B \vdash A \to B$).** Immediate from the axiom $K : B \to A \to B$ and the Deduction Theorem (Thm. 10).

5. **Disproving an implication ($\Gamma, A, \neg B \vdash \neg(A \to B)$).**
   By the true consequent property we have that $\Gamma, A, \neg B, A \to B \vdash B$ so by the Deduction Theorem (Thm. 10) $\Gamma, A, \neg B \vdash P : (A \to B) \to B$. Then:

$$\Gamma, x : A, y : \neg B \vdash \mathsf{negi}\ \underbrace{P}_{(A \to B) \to B}\ (\underbrace{K y}_{(A \to B) \to \neg B}) : \neg(A \to B)$$

6. **Conjunction introduction ($\Gamma, A, B \vdash A \land B$).** By the Deduction Theorem (Thm. 10) and the axiom pair.

7. **Negated conjunction introduction ($\Gamma, \neg A \vdash \neg(A \land B)$ and $\Gamma, \neg B \vdash \neg(A \land B)$).**

$$\Gamma, x : \neg A \vdash \mathsf{negi}\ \underbrace{\pi_1}_{(A \land B) \to A}\ (\underbrace{K x}_{(A \land B) \to \neg A}) : \neg(A \land B)$$

   Similarly:

$$\Gamma, y : \neg B \vdash \mathsf{negi}\ \underbrace{\pi_2}_{(A \land B) \to B}\ (\underbrace{K y}_{(A \land B) \to \neg B}) : \neg(A \land B)$$

8. **Disjunction introduction ($\Gamma, A \vdash A \lor B$ and $\Gamma, B \vdash A \lor B$).** By the Deduction Theorem (Thm. 10) and axioms $\mathsf{in}_1/\mathsf{in}_2$ respectively.

9. **Negated disjunction introduction ($\Gamma, \neg A, \neg B \vdash \neg(A \vee B)$).** Recall that $\vdash \mathsf{nege} : A \to \neg A \to B$, so by the Deduction Theorem (Thm. 10), we may construct $\vdash \mathsf{nege}' : \neg A \to A \to B$ for arbitrary formulas A, B.

   Now let C be any provable formula, *e.g.* $C = \top$ or $C = (z \to z)$, and let $\vdash P : C$. Then:

$$\Gamma, x : \neg A, y : \neg B \vdash \mathsf{negi}\,(\underbrace{K\,P}_{(A\vee B)\to C})\,\underbrace{(\mathsf{match}\,(\underbrace{\mathsf{nege}'\,x}_{A\to C})\,(\underbrace{\mathsf{nege}'\,y}_{B\to C}))}_{(A\vee B)\to\neg C}$$

10. **Double negation introduction ($\Gamma, A \vdash \neg\neg A$).**

$$\Gamma, x : A \vdash \mathsf{negi}\,(\underbrace{K\,x}_{\neg A\to A})\,\underbrace{\mathsf{id}}_{\neg A\to\neg A} : \neg\neg A$$

11. **Proof by cases (if $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$ then $\Gamma \vdash B$).** By the Deduction Theorem (Thm. 10) and **Contrapositive** we have that $\Gamma \vdash P : \neg B \to \neg A$ and $\Gamma \vdash Q : \neg B \to \neg\neg A$.

$$\Gamma \vdash \mathsf{dneg}\,\underbrace{(\mathsf{negi}\,P\,Q)}_{\neg\neg B} : B$$

4 Soundness and completeness

Proposition 11 (Soundness)**.** *If $\Gamma \vdash A$ then $\Gamma \vDash A$.*

Proof. By induction on the derivation of $\Gamma \vdash A$. There are three cases:

1. **Axiom.** It is routine to check that all the axioms are valid. For example, for the axiom K, let $V : \mathsf{Var} \to \mathbf{2}$ be any valuation. Then $(A \to (B \to A))^V = (1 - A^V) \sqcup (1 - B)^V \sqcup A^V = 1$.

2. **Assumption.** Let $V : \mathsf{Var} \to \mathbf{2}$ be any valuation such that $V \vDash \Gamma$. In particular $V \vDash A$ because $A \in \Gamma$ so we are done.

3. **Modus ponens.** The proof is of the form:

$$\frac{\Gamma \vdash B \to A \qquad \Gamma \vdash B}{\Gamma \vdash A}$$

   By *i.h.* we have that $\Gamma \vDash B \to A$ and $\Gamma \vDash B$. Let $V : \mathsf{Var} \to \mathbf{2}$ be any valuation such that $V \vDash \Gamma$. Then $1 = (B \to A)^V = (1 - B^V) \sqcup A^V$ and $1 = B^V$, so we must have $A^V = 1$.

□

Lemma 12. *Let A be a formula depending (at most) on the first n propositional variables $\{p_1, \ldots, p_n\}$. Let Γ be a context consisting of n formulas, the i-th of which is either p_i or $\neg p_i$. Then either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.*

Proof. By induction on A.

1. **Variable,** $A = p_i$**.** Then either $\Gamma \vdash p_i$ or $\Gamma \vdash \neg p_i$.
2. **Truth.** $\Gamma : \mathsf{trivial} : \top$.
3. **Falsity.** $\Gamma : \mathsf{negi}\ \underbrace{\mathsf{abort}}_{\bot\to\top}\ \underbrace{\mathsf{abort}}_{\bot\to\neg\top} : \neg\bot$.
4. **Negation,** $A = \neg B$**.** By *i.h.* there are two cases:
   4.1 If $\Gamma \vdash B$, by **Double negation introduction** $\Gamma \vdash \neg\neg B = \neg A$.
   4.2 If $\Gamma \vdash P : \neg B = A$ it is immediate.
5. **Conjunction,** $A = B \wedge C$**.** By *i.h.* on B there are two cases:
   5.1 If $\Gamma \vdash B$, then by *i.h.* on C there are two subcases:
   5.1.1 If $\Gamma \vdash C$, then by axiom pair we have $\Gamma \vdash B \wedge C = A$
   5.1.2 If $\Gamma \vdash \neg C$, then by **Negated conjunction introduction** $\Gamma \vdash \neg(B \wedge C) = \neg A$.
   5.2 If $\Gamma \vdash \neg B$, then by **Negated conjunction introduction** $\Gamma \vdash \neg(B \wedge C) = \neg A$.
6. **Disjunction,** $A = B \vee C$**.** By *i.h.* on B there are two cases:
   6.1 If $\Gamma \vdash B$, then by axiom in_1 we have $\Gamma \vdash B \wedge C = A$.
   6.2 If $\Gamma \vdash \neg B$, then by *i.h.* on C there are two subcases:
   6.2.1 If $\Gamma \vdash C$, then by axiom in_2 we have $\Gamma \vdash B \vee C = A$
   6.2.2 If $\Gamma \vdash \neg C$, then by **Negated disjunction introduction** $\Gamma \vdash \neg(B \vee C) = \neg A$.
7. **Implication,** $A = B \to C$**.** By *i.h.* on B there are two cases:
   7.1 If $\Gamma \vdash B$, then by *i.h.* on C there are two cases:
   7.1.1 If $\Gamma \vdash C$, then by **True consequent** we have $\Gamma \vdash B \to C = A$.
   7.1.2 If $\Gamma \vdash \neg C$, then by **Disproving an implication** we have $\Gamma \vdash \neg(B \to C) = A$.
   7.2 If $\Gamma \vdash \neg B$, then by **False antecedent** we have $\Gamma \vdash B \to C = A$.

□

The previous lemma can be generalized when the context consists of $k \leq n$ formulas.

Lemma 13. *Let A be a formula depending (at most) on the first n propositional variables $\{p_1, \ldots, p_n\}$. Let Γ be a context consisting of $k \leq n$ formulas, the i-th of which is either p_i or $\neg p_i$. Moreover, suppose that $\models A$. Then either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.*

Proof. By induction on k downwards from n to 0 (*i.e.* on $n - k$).

1. **Base case,** $k = n$**.** This is precisely the previous lemma (Lem. 12).

2. **Induction, "$k+1 \implies k$".** Consider the context Γ extended with p_{k+1}. By *i.h.* there are two possibilities, $\Gamma, p_{k+1} \vdash A$ or $\Gamma, p_{k+1} \vdash \neg A$. The second case is not possible since, by Soundness (Prop. 11), we would have that $\Gamma, p_{k+1} \vDash \neg A$, *i.e.* there is a valuation V such that $V \vDash \Gamma, p_{k+1}$ and $A^V = 0$. But note that $\vDash A$ by hypothesis, so $A^V = 1$, a contradiction. Hence $\Gamma, p_{k+1} \vdash A$.

   Similarly, if we consider the context Γ extended with $\neg p_{k+1}$ we obtain by *i.h.* that $\Gamma, \neg p_{k+1} \vdash A$.

   Finally, note that $\Gamma, p_{k+1} \vdash A$ and $\Gamma, \neg p_{k+1} \vdash A$ together entail $\Gamma \vdash A$ using the principle of **Proof by cases**.

□

Theorem 14 (Completeness)**.** *The following hold:*

1. **Completeness.** *If* $\vDash A$ *then* $\vdash A$.
2. **Implicational completeness.** *If* $\Gamma \vDash A$ *then* $\Gamma \vdash A$.

Proof. For item 1., by the previous lemma (Lem. 13) in the particular case in which $k = 0$, we have that either $\vdash A$ or $\vdash \neg A$. The second case is not possible since, by Soundness (Prop. 11), we would have that $\vDash \neg A$, *i.e.* that there is a valuation V such that $A^V = 0$. But note that $\vDash A$ by hypothesis, so $A^V = 1$, a contradiction. Hence $\vdash A$.

For item 2., suppose that $\Gamma \vDash A$. By the corollary of compactness (Coro. 6) there exists a finite subset $\{B_1, \dots, B_n\} \subseteq \Gamma$ such that $B_1, \dots, B_n \vDash A$. This in turn means that $\vDash B_1 \to \dots \to B_n \to A$. By item 1. of this theorem, we have $\vdash B_1 \to \dots \to B_n \to A$. By the Deduction Theorem (Thm. 10) we have $\Gamma' \vdash A$. Finally, by Weakening (Lem. 9), $\Gamma \vdash A$. □