Sequent Calculus and Resolution for Propositional Logic

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1 Sequent Calculus

1.1 Proof system

Definition 1 (Formulas)**.** The set of formulas is given by:

$$A ::= p \mid \neg A \mid A \vee A \mid A \wedge A \mid A \to A$$

Recall that a formula is *valid* (or a *tautology*) if every valuation verifies it, and satisfiable if there is a valuation that verifies it.

Definition 2 (Sequents)**.** A *sequent* is a judgment of the form $A_1, \dots, A_n \vdash B_1, \dots, B_m$. A sequent is *valid* if $\left(\bigwedge_{i=1}^{n} A_i\right) \to \left(\bigvee_{i=1}^{m} B_i\right)$ is valid.

Definition 3. The *propositional sequent calculus* LK is given by the following rules:

Structural rules (weak)

Exchange
$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ \texttt{EL} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ \texttt{ER}$$

Contraction
$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \texttt{CL} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \texttt{CR}$$

Weakening
$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \texttt{WL} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \texttt{WR}$$

Axiom and cut rule (strong)

$$\frac{}{A \vdash A}\ \texttt{ax} \qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \texttt{cut}$$

Rules for propositional connectives (strong)

Negation
$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg\mathrm{L} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg\mathrm{R}$$

Conjunction
$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge\mathrm{L} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \wedge\mathrm{R}$$

Disjunction
$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \vee\mathrm{L} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \vee\mathrm{R}$$

Implication
$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ {\to}\mathrm{L} \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ {\to}\mathrm{R}$$

Remark 4. As an alternative to the right rule for disjunction $\vee\mathrm{R}$, one may define the propositional sequent calculus LK' with the two following rules:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ \vee\mathrm{R}_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ \vee\mathrm{R}_2$$

It can be readily checked that $\{\vee\mathrm{R}\}$ and $\{\vee\mathrm{R}_1, \vee\mathrm{R}_2\}$ are interderivable:

$$\frac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A, A \vee B}\ \vee\mathrm{R}_2}{\Gamma \vdash \Delta, A \vee B, A}\ \mathtt{ER}}{\Gamma \vdash \Delta, A \vee B, A \vee B}\ \vee\mathrm{R}_1}{\Gamma \vdash \Delta, A \vee B}\ \mathtt{CR} \qquad \frac{\dfrac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A, B}\ \mathtt{WR}}{\Gamma \vdash \Delta, A \vee B}\ \vee\mathrm{R} \qquad \frac{\dfrac{\dfrac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, B, A}\ \mathtt{WR}}{\Gamma \vdash \Delta, A, B}\ \mathtt{ER}}{\Gamma \vdash \Delta, A \vee B}\ \vee\mathrm{R}$$

Proposition 5 (Subformula property). *If π is a* cut*-free proof, every formula in π is a subformula of the formula in the end sequent.*

Proof. Observe that, in every rule other than cut, formulas in the upper sequents are subformulas of formulas in the lower sequent. □

Definition 6. The *length* of a proof π, written $\|\pi\|$, is the number of *strong inferences* in π (excluding exchange, contraction, weakening, and axioms).

Definition 7. A sequent $\Gamma \vdash \Delta$ is called *classical* (resp. *intuitionistic*, *minimal*, *Pierce type*) according to the following table. The proof system restricted to sequents of the given form is called LK (resp. LJ, LM, LP).

classical	(without restrictions)	(LK)
intuitionistic	Δ has at most one formula	(LJ)
minimal	Δ has exactly one formula	(LM)
Pierce type	Δ has at least one formula	(LP)

(**Important:** in the intuitionistic and minimal variants, rules $\vee\mathrm{R}_1$ and $\vee\mathrm{R}_2$ rather than $\vee\mathrm{R}$ should be used.)

Example 8. Items 1–11 are the Hilbert-style axioms. Items 13–16 are various De Morgan laws.

1. S: $A \to (B \to A)$
2. K: $(A \to B \to C) \to (A \to B) \to A \to C$
3. negi: $(A \to B) \to (A \to \neg B) \to \neg A$
4. nege: $A \to (\neg A \to B)$
5. dneg: $\neg\neg A \to A$ . . . . strictly classical
6. pair: $A \to B \to (A \wedge B)$
7. π_1: $(A \wedge B) \to A$
8. π_2: $(A \wedge B) \to B$
9. in_1: $A \to (A \vee B)$
10. in_2: $B \to (A \vee B)$
11. match: $(A \to C) \to (B \to C) \to (A \vee B) \to C$
12. De Morgan: $\neg(p \vee q) \to (\neg p \wedge \neg q)$
13. De Morgan: $(\neg p \wedge \neg q) \to \neg(p \vee q)$
14. De Morgan: $(\neg p \vee \neg q) \to \neg(p \wedge q)$
15. De Morgan: $\neg(p \wedge q) \to (\neg p \vee \neg q)$ . . . . strictly classical
16. Contrapositive: $(p \to q) \to (\neg q \to \neg p)$ . . . . strictly classical
17. Pierce's law: $((p \to q) \to p) \to p$ . . . . strictly classical

1.2 Soundness and completeness

Proposition 9 (Soundness)**.** *The propositional sequent calculus is sound, i.e. provable sequents are valid.*

Proof. Observe that every rule preserves the property that the sequents are tautologies. □

Lemma 10 (Inversion)**.** *Let I be an inference other than weakening. If the lower sequent is valid then the upper sequents are valid.*

Proof. Straightforward by inspection on all the rules except for `WL` and `WR`. □

Lemma 11 (Completeness, with bounds)**.** *Let $\Gamma \vdash \Delta$ be a valid sequent with m logical connectives. Then there is a* `cut`*-free proof with strictly less than 2^m strong inferences.*

Proof. By induction on m. If $m = 0$, every formula is a propositional variable, so there is a variable p occurring in both Γ and Δ. Hence we may prove $\Gamma \vdash \Delta$ from the axiom $p \vdash p$ (using no strong inferences).

Now let $m > 0$. Note there is a formula either in Γ or in Δ whose outermost connective is either $\neg$, $\wedge$, $\vee$, or $\to$. Hence there are eight subcases:

1. **Left negation.** Suppose that there is a formula $\neg A$ in Γ, and let Γ' be the result of removing $\neg A$ from Γ. Note that $\neg A, \Gamma' \vdash \Delta$ is valid so by Inversion (Lem. 10) we have that $\Gamma' \vdash \Delta, A$ is also valid and uses strictly less than m connectives. So by *i.h.* there is a proof π of $\Gamma' \vdash \Delta, A$ using strictly less than 2^{m-1} strong inferences, and we may construct a proof π':

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma' \vdash \Delta, A}}{\neg A, \Gamma' \vdash \Delta}\ \neg\mathrm{L}}{\Gamma \vdash \Delta}$$

   Then $\|\pi'\| < 2^{m-1} + 1 \leq 2^m$.

2. **Right negation.** Similar to the **Left negation** case.

3. **Left conjunction.** Suppose that there is a formula $A \wedge B$ in Γ, and let Γ' be the result of removing $A \wedge B$ from Γ. Note that $A \wedge B, \Gamma' \vdash \Delta$ is valid so by Inversion (Lem. 10) we have that $A, B, \Gamma' \vdash \Delta$ is also valid and uses strictly less than m connectives. So by *i.h.* there is a proof π of $A, B, \Gamma' \vdash \Delta$ using strictly less than 2^{m-1} strong inferences, and we may construct a proof π':

$$\dfrac{\dfrac{\dfrac{\pi}{A, B, \Gamma' \vdash \Delta}}{A \wedge B, \Gamma' \vdash \Delta}\ \wedge\mathrm{L}}{\Gamma \vdash \Delta}$$

   Then $\|\pi'\| < 2^{m-1} + 1 \leq 2^m$.

4. **Right conjunction.** Suppose that there is a formula AB in Δ, and let Δ' be the result of removing $A \wedge B$ from Δ. Note that by Inversion (Lem. 10) the sequents $\Gamma \vdash \Delta', A$ and $\Gamma \vdash \Delta', B$ are both valid, and use strictly less than m connectives. So by *i.h.* there is a proof π_1 of $\Gamma \vdash \Delta', A$ and a proof π_2 of $\Gamma \vdash \Delta', B$, each using strictly less than 2^{m-1} strong inferences, and we may construct a proof π':

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta', A} \qquad \dfrac{\pi_2}{\Gamma \vdash \Delta', B}}{\Gamma \vdash \Delta', A \wedge B}\ \wedge\mathrm{R}}{\Gamma \vdash \Delta}$$

   Then $\|\pi\| = 1 + \|\pi_1\| + \|\pi_2\| < 2^{m-1}$.

5. **Left disjunction.** Similar to the **Right conjunction** case.

6. **Right disjunction.** Similar to the **Left conjunction** case.

7. **Left implication.** Similar to the **Right conjunction** case.

8. **Right implication.** Similar to the **Left conjunction** case.

$\square$

Theorem 12 (Completeness). *Let $\Gamma \vdash \Delta$ be a valid sequent. Then it has a* `cut`*-free proof.*

Proof. An immediate consequence of Lem. 11. □

Theorem 13 (Cut elimination). *Let $\Gamma \vdash \Delta$ be a provable sequent. Then there is a* `cut`*-free proof of $\Gamma \vdash \Delta$.*

Proof. An immediate consequence of Soundness (Prop. 9) and Completeness (Thm. 12). □

2 Refutation by Propositional Resolution

Definition 14. A *literal* is a propositional variable p or its negation $\neg p$. For literals we define the convolution $\overline{p} \stackrel{\text{def}}{=} \neg p$ and $\overline{\neg p} \stackrel{\text{def}}{=} p$. A *clause* is a finite set of literals. A clause $\mathscr{C} = \{x_1, \dots, x_n\}$ represents, informally, the disjunction $\bigvee_{i=1}^{n} x_i$.

If $\mathscr{C}$ and $\mathscr{D}$ are clauses and x is a literal such that $x \in \mathscr{C}$ and $\overline{x} \in \mathscr{D}$, we define the *resolvent of $\mathscr{C}$ and $\mathscr{D}$ (with respect to x)* as:

$$\mathsf{R}_x(\mathscr{C}, \mathscr{D}) \stackrel{\text{def}}{=} (\mathscr{C} \setminus \{x\}) \cup (\mathscr{D} \setminus \{\overline{x}\})$$

A set of clauses $\Gamma = \{\mathscr{C}_i\}_{i \in I}$ represents, informally, the conjunction $\bigwedge_{i \in I} \mathscr{C}_i$. Given a set of clauses Γ and a clause $\mathscr{C}$, we define a judgment $\Gamma \rhd \mathscr{C}$ whose meaning is defined via the following system of *Propositional Resolution*:

$$\frac{\mathscr{C} \in \Gamma}{\Gamma \rhd \mathscr{C}} \qquad \frac{\Gamma \rhd \mathscr{C} \quad \Gamma \rhd \mathscr{D} \quad x \in \mathscr{C} \quad \overline{x} \in \mathscr{D}}{\Gamma \rhd \mathsf{R}_x(\mathscr{C}, \mathscr{D})}$$

The second rule is called the *resolution rule*. A *refutation* of Γ is a derivation ending in $\Gamma \rhd \varnothing$.

Proposition 15 (Correctness of Propositional Resolution). *If $\Gamma \rhd \varnothing$ then Γ is unsatisfiable.*

Proof. We prove the two following claims:

1. Let Γ be a satisfiable context and let $\mathscr{C}$ be the resolvent of two clauses in Γ. Then $\Gamma \cup \{\mathscr{C}\}$ is satisfiable.
2. Suppose that $\Gamma \rhd \mathscr{C}$ is derivable, and let $\Gamma \subseteq \Delta$ where Δ is satisfiable. Then $\Delta \cup \{\mathscr{C}\}$ is satisfiable.

To prove the first claim, suppose that $\mathscr{C} = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ is the resolvent of $\{x_1, \dots, x_n, z\} \in \Gamma$ and $\{y_1, \dots, y_m, \overline{z}\} \in \Gamma$. Let V be a valuation that verifies Γ. If $z^V = 0$, then V must verify at least one of $x_1, \dots, x_m$ so $\mathscr{C}^V = 1$. On the other hand, if $z^V = 1$ then $\overline{z}^V = 0$ so V must verify at least one of $y_1, \dots, y_m$ and also $\mathscr{C}^V = 1$.

To prove the second claim, proceed by induction on the derivation of $\Gamma \rhd \mathscr{C}$. For the axiom we have that $\mathscr{C} \in \Gamma \subseteq \Delta$ and Δ is satisfiable by hypothesis so indeed $\Delta \cup \{\mathscr{C}\} = \Delta$ is satisfiable. For the resolution rule, suppose that we deduce $\Gamma \rhd \mathsf{R}_x(\mathscr{C}, \mathscr{D})$ from $\Gamma \rhd \mathscr{C}$ and $\Gamma \rhd \mathscr{D}$, and let $\Gamma \subseteq \Delta$ such that Δ is satisfiable. By *i.h.* on the first premise we obtain that $\Delta \cup \{\mathscr{C}\}$ is satisfiable. Now since $\Gamma \subseteq \Delta \cup \{\mathscr{C}\}$ we may apply the *i.h.* on the second premise to obtain that $\Delta \cup \{\mathscr{C}, \mathscr{D}\}$ is satisfiable. Finally by the first claim we have that $\Delta \cup \{\mathscr{C}, \mathscr{D}, \mathsf{R}_x(\mathscr{C}, \mathscr{D})\}$ is satisfiable. So $\Delta \cup \{\mathsf{R}_x(\mathscr{C}, \mathscr{D})\}$ is also satisfiable.

Now to conclude the proof of the proposition, let $\Gamma \rhd \varnothing$, and suppose that Γ is satisfiable. Then by the second claim we have that $\Gamma \cup \{\varnothing\}$ is satisfiable, which is a contradiction. Hence Γ is unsatisfiable. □

Remark 16. One can apply the resolution method to check whether a formula A is valid. More precisely, note that A is valid if $\neg A$ is unsatisfiable. Rewrite $\neg A$ as an equivalent formula in conjunctive normal form $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} x_{ij}$ and apply the resolution method to the set $\{\mathscr{C}_i \mid 1 \leq i \leq n\}$ where $\mathscr{C}_i$ is the clause $\mathscr{C}_i = \{x_{ij} \mid 1 \leq j \leq m_i\}$.

Example 17. To check that $\neg(p \wedge q) \to (\neg p \vee \neg q)$ is valid, consider its negation $\neg(\neg(p \wedge q) \to (\neg p \vee \neg q))$ and note that it is equivalent to $p \wedge q \wedge (\neg p \vee \neg q)$. Then taking $\Gamma = \{\{p\}, \{q\}, \{\neg p, \neg q\}\}$ one has:

$$\dfrac{\Gamma \rhd \{p\} \qquad \dfrac{\dfrac{}{\Gamma \rhd \{q\}} \qquad \dfrac{}{\Gamma \rhd \{\neg p, \neg q\}}}{\Gamma \rhd \{\neg p\}}}{\Gamma \rhd \varnothing}$$

Theorem 18 (Completeness of Propositional Resolution)**.** *If Γ is unsatisfiable, then $\Gamma \rhd \varnothing$ is derivable.*

Proof. By Compactness, we may assume that Γ is finite. Hence the number n of distinct variables that occur anywhere in Γ is finite. We proceed by induction on n. If $n = 0$, note that Γ cannot be empty, for it would be satisfiable, so $\Gamma = \{\varnothing\} \rhd \varnothing$ and we are done.

If $n > 0$, let p be one variable that occurs somewhere in Γ. If a clause $\mathscr{C} \in \Gamma$ contains both p and $\neg p$, then $\mathscr{C}$ is trivially satisfiable (*i.e.* any valuation verifies it) so without loss of generality we may assume that Γ has no such clauses. Now let:

$$\begin{array}{rcl} \Delta & \stackrel{\text{def}}{=} & \{\mathsf{R}_p(\mathscr{C}, \mathscr{D}) \mid \mathscr{C} \in \Gamma,\ \mathscr{D} \in \Gamma,\ p \in \mathscr{C},\ \overline{p} \in \mathscr{D}\} \\ & \cup & \{\mathscr{C} \mid \mathscr{C} \in \Gamma,\ x \notin \mathscr{C},\ \overline{x} \notin \mathscr{C}\} \end{array}$$

To conclude, observe that Δ is satisfiable if and only if Γ is satisfiable, and the variable p does not occur in Δ, so by *i.h.* we have that $\Delta \rhd \varnothing$ as required. $\square$