Minimal CPS Translation

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2019-11-26

Definition 1 (Syntax).

$$\begin{array}{rcl} t & \stackrel{\text{def}}{=} & x \mid \lambda x.\, t \mid t\, t \\ v & \stackrel{\text{def}}{=} & \lambda x.\, t \end{array}$$

Definition 2 (Call-by-name — weak, closed, small-step semantics).

$$\frac{}{(\lambda x.\, t)\, s \to t\{x := s\}}\ \beta \qquad \frac{t \to t'}{t\, s \to t'\, s}\ \mu$$

Definition 3 (Call-by-value — weak, closed, big-step semantics).

$$\frac{}{\lambda x.\, t \Downarrow \lambda x.\, t}\ \texttt{e-lam} \qquad \frac{t \Downarrow \lambda x.\, t' \quad s \Downarrow v \quad t'\{x := v\} \Downarrow v'}{t\, s \Downarrow v'}\ \texttt{e-app}$$

Definition 4 (CPS translation).

$$\begin{array}{rcl} x^\circ & \stackrel{\text{def}}{=} & \lambda k.\, k\, x \\ (\lambda x.\, t)^\circ & \stackrel{\text{def}}{=} & \lambda k.\, k\, (\lambda x.\, t)^\bullet \\ (t\, s)^\circ & \stackrel{\text{def}}{=} & \lambda k.\, t^\circ\, (\lambda x.\, s^\circ\, (\lambda y.\, x\, y\, k)) \\ \\ (\lambda x.\, t)^\bullet & \stackrel{\text{def}}{=} & \lambda x.\, t^\circ \end{array}$$

Lemma 5. $t\{x := v\}^\circ = t^\circ\{x := v^\bullet\}$

Proof. By induction on t. □

Theorem 6. *If $t \Downarrow v$ then $t^\circ\ K \twoheadrightarrow K\ v^\bullet$ for any term K.*

Proof. By induction on the derivation of $t \Downarrow v$.

- `e-lam`:

$$(\lambda x.\, t)^\circ\ k = (\lambda k.\, k\, (\lambda x.\, t)^\bullet)\, K \to K\, (\lambda x.\, t)^\bullet$$

- `e-app`:

$$\begin{array}{rcll} (t\, s)^\circ\ K & \stackrel{\text{def}}{=} & (\lambda k.\, t^\circ\, (\lambda x.\, s^\circ\, (\lambda y.\, x\, y\, k)))\, K & \\ & \to & t^\circ\, (\lambda x.\, s^\circ\, (\lambda y.\, x\, y\, K)) & \\ & \twoheadrightarrow & (\lambda x.\, s^\circ\, (\lambda y.\, x\, y\, K))\, (\lambda x.\, t')^\bullet & \text{by } \textit{i.h.} \text{ on the first premise} \\ & \to & s^\circ\, (\lambda y.\, (\lambda x.\, t')^\bullet\, y\, K) & \\ & \twoheadrightarrow & (\lambda y.\, (\lambda x.\, t')^\bullet\, y\, K)\, v^\bullet & \text{by } \textit{i.h.} \text{ on the second premise} \\ & \to & (\lambda x.\, t')^\bullet\, v^\bullet\, K & \\ & \to & t'^\circ\{x := v^\bullet\}\, K & \\ & = & t'\{x := v\}^\circ\, K & \text{by Lem. 5} \\ & \twoheadrightarrow & K\, v'^\bullet & \text{by } \textit{i.h.} \text{ on the third premise} \end{array}$$

□

1 Types

Definition 7 (Simply typed λ-calculus).

$$\begin{array}{lcl} A & ::= & \bot \mid \alpha \mid A \to A \\ \Gamma & ::= & \varnothing \mid \Gamma, x : A \end{array} \qquad \neg A \stackrel{\text{def}}{=} A \to \bot$$

Rules:

$$\dfrac{}{\Gamma, x : A \vdash x : A}\,\texttt{t-var} \qquad \dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B}\,\texttt{t-lam} \qquad \dfrac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash t\, s : B}\,\texttt{t-app}$$

Definition 8 (CPS translation on types).

$$\bot^\circ \stackrel{\text{def}}{=} \bot \qquad \alpha^\circ \stackrel{\text{def}}{=} \alpha \qquad (A \to B)^\circ \stackrel{\text{def}}{=} A^\circ \to \neg\neg B^\circ$$

$$\varnothing^\circ \stackrel{\text{def}}{=} \varnothing \qquad (\Gamma, x : A)^\circ \stackrel{\text{def}}{=} \Gamma^\circ, x : A^\circ$$

Theorem 9. *If* $\Gamma \vdash t : A$ *then* $\Gamma^\circ \vdash t^\circ : \neg\neg A^\circ$.

Proof. By induction on the derivation of $\Gamma \vdash t : A$.

- `t-var`:

$$\dfrac{\dfrac{\dfrac{}{\Gamma^\circ, x : A^\circ, k : \neg A^\circ \vdash k : \neg A^\circ} \qquad \dfrac{}{\Gamma^\circ, x : A^\circ, k : \neg A^\circ \vdash x : A^\circ}}{\Gamma^\circ, x : A^\circ, k : \neg A^\circ \vdash k\, x : \bot}}{\Gamma^\circ, x : A^\circ \vdash x^\circ = \lambda k.\, k\, x : \neg\neg A^\circ}$$

- `t-lam`:

$$\dfrac{\dfrac{\dfrac{}{\Gamma^\circ, k : \neg(A^\circ \to \neg\neg B^\circ) \vdash k : \neg(A^\circ \to \neg\neg B^\circ)} \qquad \dfrac{\dfrac{\textit{i.h.} + \text{weakening}}{\Gamma^\circ, k : \neg(A^\circ \to \neg\neg B^\circ), x : A^\circ \vdash t^\circ : \neg\neg B^\circ}}{\Gamma^\circ, k : \neg(A^\circ \to \neg\neg B^\circ) \vdash \lambda x.\, t^\circ : A^\circ \to \neg\neg B^\circ}}{\Gamma^\circ, k : \neg(A^\circ \to \neg\neg B^\circ) \vdash k\,(\lambda x.\, t^\circ) : \bot}}{\Gamma^\circ \vdash (\lambda x.\, t)^\circ = \lambda k.\, k\,(\lambda x.\, t^\circ) : \neg\neg(A^\circ \to \neg\neg B^\circ)}$$

- `t-app`:

$$\dfrac{\dfrac{\dfrac{\textit{i.h.} + \text{weakening}}{\Gamma^\circ, k : \neg B^\circ \vdash t^\circ : \neg\neg(A^\circ \to \neg\neg B^\circ)} \qquad \dfrac{\dfrac{\dfrac{\textit{i.h.} + \text{weakening}}{\Delta \vdash s^\circ : \neg\neg A^\circ} \qquad \dfrac{\text{by } \pi \text{ below}}{\Delta \vdash \lambda y.\, x\, y\, k : \neg A^\circ}}{\underbrace{\Gamma^\circ, k : \neg B^\circ, x : A^\circ \to \neg\neg B^\circ}_{\Delta} \vdash s^\circ(\lambda y.\, x\, y\, k) : \bot}}{\Gamma^\circ, k : \neg B^\circ \vdash \lambda x.\, s^\circ(\lambda y.\, x\, y\, k) : \neg(A^\circ \to \neg\neg B^\circ)}}{\Gamma^\circ, k : \neg B^\circ \vdash t^\circ(\lambda x.\, s^\circ(\lambda y.\, x\, y\, k)) : \bot}}{\Gamma^\circ \vdash (t\, s)^\circ = \lambda k.\, t^\circ(\lambda x.\, s^\circ(\lambda y.\, x\, y\, k)) : \neg\neg B^\circ}$$

  where π is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, y : A^\circ \vdash x : A^\circ \to \neg\neg B^\circ} \qquad \dfrac{}{\Delta, y : A^\circ \vdash y : A^\circ}}{\Delta, y : A^\circ \vdash x\, y : \neg\neg B^\circ} \qquad \dfrac{}{\Delta, y : A^\circ \vdash k : \neg B^\circ}}{\Delta, y : A^\circ \vdash x\, y\, k : \bot}}{\Delta \vdash \lambda y.\, x\, y\, k : \neg A^\circ}$$

□