REESCRITURA $\quad \vdash M =_\beta N \quad\quad x\cdot(y\cdot z) \to (x\cdot y)\cdot z$

DEF. Un sistema de reescritura abstracto (ARS) es un par $(A, \to)$ donde A es un cjto. de "objetos", y $\to \subseteq A^2$ es una relación de "reducción". "$a \to b$"

DEF. 1) Un objeto $a \in A$ es weakly normalizing (WN) si $\exists b \in A.\ a \twoheadrightarrow b$ y $\underbrace{b \text{ es f.n.}}_{\Updownarrow}$

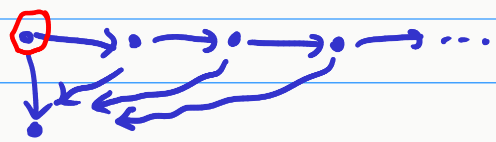
$\nexists c.\ b \to c$

2) Un objeto $a \in A$ es Strongly normalizing (SN) si $\nexists a_1, a_2, \dots .\ a \to a_1 \to a_2 \to a_3 \to \dots$

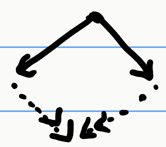
OBS. • a SN $\Rightarrow$ a WN

Si a No es WN, entonces a no es una f.n., entonces $a \to a'$ y a' No es WN

OBS. a WN $\not\Rightarrow$ a SN.



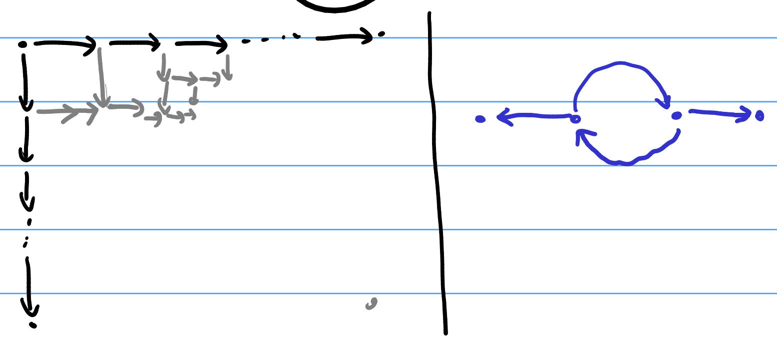
3) Un objeto $a \in A$ es weakly Church-Rosser (WCR) si $\forall b, c \in A$ tq. $a \to b \quad a \to c$
$\exists d \in A$ tq. $b \twoheadrightarrow d \quad c \twoheadrightarrow d$



4) Un objeto $a \in A$ es Church-Rosser (CR) si $\forall b, c \in A.$ tq. $a \twoheadrightarrow b \quad a \twoheadrightarrow c$
$\exists d \in A$ tq. $b \twoheadrightarrow d \quad c \twoheadrightarrow d$

OBS.

- a CR $\Rightarrow$ a WCR
- WCR $\not\Rightarrow$ CR



¿Por qué son importantes?

$$\twoheadleftarrow\!\!\twoheadrightarrow \;=\; (\to \cup \leftarrow)^*$$

Lema. Si un sistema es CR y $a \twoheadleftarrow\!\!\twoheadrightarrow b$, existe un c tq. $a \twoheadrightarrow c$, $b \twoheadrightarrow c$

Dem. Inducción en la cantidad de pasos de reducción/expansión.

1) Si hay 0 pasos, $(a = b)$. $c := a = b$.

2) Si hay al menos un paso,

2.1) Reducción:

$$\begin{tikzcd}
a \arrow[r] & a' \arrow[r, leftrightarrow, "*"] \arrow[dr, twoheadrightarrow] & b \arrow[d, twoheadrightarrow] \arrow[dl, phantom, "HI"] \\
& & c
\end{tikzcd}$$

2.2) Expansión:

$$\begin{tikzcd}
a \arrow[ddr, twoheadrightarrow] & a' \arrow[l] \arrow[r, leftrightarrow, "*"] \arrow[dr, twoheadrightarrow] \arrow[dl, phantom, "CR"] & b \arrow[d, twoheadrightarrow] \arrow[dl, phantom, "HI"] \\
& & c \arrow[dl, twoheadrightarrow] \\
& c' &
\end{tikzcd}$$

Lema. Si un sistema es SN y CR entonces son equiv:

1) $a \twoheadleftarrow\!\!\twoheadrightarrow b$

2) $\exists c$ f.n. tq. $a \twoheadrightarrow c \twoheadleftarrow b$

Dem. $2 \Rightarrow 1$ ✓ $\quad 1 \Rightarrow 2$

$$\begin{tikzcd}
a \arrow[d, leftrightarrow, "*"] \arrow[dr, twoheadrightarrow] & & \\
b \arrow[r, twoheadrightarrow, "CR"'] & c \arrow[r, twoheadrightarrow, "SN"] & c' \text{ f.n}
\end{tikzcd}$$

Obs. Si un sist. es CR, un objeto a no tiene más de una f.n.

o sea si
$$\begin{tikzcd} a \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "CR"] & b \arrow[d, dotted, two heads] \\ c \arrow[r, dotted, two heads] & {} \end{tikzcd}$$
b, c f.n.s

$b = c$

Lema. (Newman) $SN \wedge WCR \Rightarrow CR$.

Dem. Un objeto a se dice ambiguo si reduce a dos f.n.s distintas, o sea $a \twoheadrightarrow b$, $a \twoheadrightarrow c$, b, c f.n.s $b \neq c$.

CLAIM: Si un objeto a es ambiguo $\exists a'.\ a \to a'$, a' ambiguo.

Dem.
$$\begin{tikzcd} & a_1 \arrow[rr, two heads] \arrow[dr] & & b \\ a \arrow[ur] \arrow[dr] \arrow[rr, phantom, "WCR"] & & a_3 \arrow[r, two heads, "SN"] & d \\ & a_2 \arrow[ur] \arrow[rr, two heads] & & c \end{tikzcd}$$

• $b \neq c$

VALE: $b \neq d \vee d \neq c$

Sea
$$\begin{tikzcd} & b \arrow[r, two heads, "SN"] & b' \arrow[d, equal] \\ a \arrow[ur] \arrow[dr] & & {} \\ & c \arrow[r, two heads, "SN"] & c' \end{tikzcd}$$

$b' = c'$

$$\begin{tikzcd}[column sep=small] & I(Iz) \arrow[dl] \arrow[dr] & \\ Iz \arrow[dr] & & Iz \arrow[dl] \\ & z & \end{tikzcd}$$

Cálculo-λ:

Sintaxis: $t ::= x \mid \lambda x.t \mid ts$

Semántica:

$$\frac{}{(\lambda x.t)s \to t\{x:=s\}}\,\beta \qquad \frac{t \to t'}{ts \to t's}\,\mu \qquad \frac{s \to s'}{ts \to ts'}\,\nu$$

$t[x/s]$

$$\frac{t \to t'}{\lambda x.t \to \lambda x.t'}\,\xi$$

ALT: $C[(\lambda x.t)s] \to C[t\{x:=s\}]$

- CL no es WN (y mucho menos SN).

$$\Omega = (\lambda x.xx)(\lambda x.xx) \to \Omega \to \Omega \to \Omega \dots$$

- Hay términos que son WN y no son SN:

$$\begin{tikzcd} (\lambda x.y)\Omega \arrow[r] \arrow[d] & (\lambda x.y)\Omega \arrow[r] \arrow[dl] & (\lambda x.y)\Omega \arrow[r] \arrow[dll] & \dots \\ y & & & \end{tikzcd}$$

Lema. CL es WCR.

Dem. Sea t un término.
Procedemos por inducción en t:

1) $t = x$. ✓

2) $t = \lambda x.t'$, Supongamos que

$$\begin{tikzcd}[column sep=small] & \lambda x.t' \arrow[dl] \arrow[dr] & \\ t_1 = \lambda x.t_1' \arrow[dr] & & t_2 = \lambda x.t_2' \arrow[dl] \\ & \lambda x.t_3' & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & t' \arrow[dl] \arrow[dr] & \\ t_1' \arrow[dr] & \text{HI} & t_2' \arrow[dl] \\ & t_3' & \end{tikzcd}$$

"Inmediato por HI".

3) $t = su$.

$$\begin{tikzcd} & su \arrow[dl] \arrow[dr] & \\ r_1 & & r_2 \end{tikzcd}$$

μ vs. μ.

$$\begin{tikzcd} & su \arrow[dl] \arrow[dr] & \\ s_1 u \arrow[dr] & & s_2 u \arrow[dl] \\ & s_3 u & \end{tikzcd} \qquad \begin{tikzcd} & s \arrow[dl] \arrow[dr] & \\ s_1 \arrow[dr] \arrow[rr, phantom, "HI"] & & s_2 \arrow[dl] \\ & s_3 & \end{tikzcd}$$

μ vs. ν.
"ORTOGONALES"

$$\begin{tikzcd} & su \arrow[dl] \arrow[dr] & \\ s'u \arrow[dr] & & su' \arrow[dl] \\ & s'u' & \end{tikzcd} \qquad \begin{array}{l} s \to s' \\ u \to u' \end{array}$$

ν vs. ν.

$$\begin{tikzcd} & su \arrow[dl] \arrow[dr] & \\ su_1 \arrow[dr] & & su_2 \arrow[dl] \\ & su_3 & \end{tikzcd} \qquad \begin{tikzcd} & u \arrow[dl] \arrow[dr] & \\ u_1 \arrow[dr] \arrow[rr, phantom, "HI"] & & u_2 \arrow[dl] \\ & u_3 & \end{tikzcd}$$

β vs. β.

$$\begin{tikzcd} & (\lambda x.t)s \arrow[dl] \arrow[dr] & \\ t\{x:=s\} \arrow[rr, phantom, "="] & & t\{x:=s\} \end{tikzcd}$$

β vs. μ.

$$\begin{tikzcd} & (\lambda x.t)s \arrow[dl] \arrow[dr] & \\ t\{x:=s\} \arrow[dr] & & (\lambda x.t')s \arrow[dl] \\ & t'\{x:=s\} & \end{tikzcd} \qquad \begin{array}{l} \lambda x.t \to \lambda x.t' \\ t \to t' \end{array}$$

LEMA: Si $t \to t'$ entonces $t\{x:=s\} \to t'\{x:=s\}$.
(Dem. por inducción en t).

β vs. ν.

$s \to s'$

$$\begin{tikzcd}
 & (\lambda x.t)\,s \arrow[dl] \arrow[dr] & \\
t\{x:=s\} \arrow[drr, twoheadrightarrow] & & (\lambda x.t)\,s' \arrow[d] \\
 & & t\{x:=s'\}
\end{tikzcd}$$

Lema: Si $s \to s'$ entonces $t\{x:=s\} \twoheadrightarrow t\{x:=s'\}$. $\square$

$$\begin{tikzcd}
 & (\lambda x.y)\,s \arrow[dl] \arrow[dr] & \\
y \arrow[dr, equal] & & (\lambda x.y)\,s' \arrow[dl] \\
 & y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\lambda x.xx)\,s \arrow[dl] \arrow[dr] & \\
ss \arrow[d] \arrow[dr] & & (\lambda x.xx)\,s' \arrow[ddl] \\
s's \arrow[dr] & ss' \arrow[d] & \\
 & s's' &
\end{tikzcd}$$

Conclusión (?). CL es WCR,
pero la demo de arriba no alcanza
para probar CR (porque CL no es SN).

Teorema. CL es CR.

- T+M-L
- Finite developments
- Etiquetas de Lévy
- Z property.

IDEA. Usar la técnica de Tait — Martin-Löf.

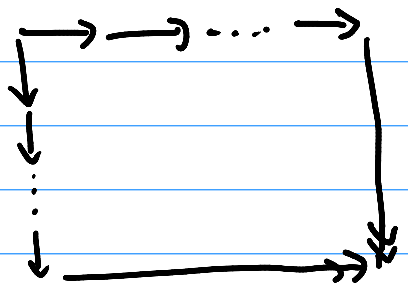
Definir una relación $\Rightarrow \subseteq \Lambda^2$ que verifique:

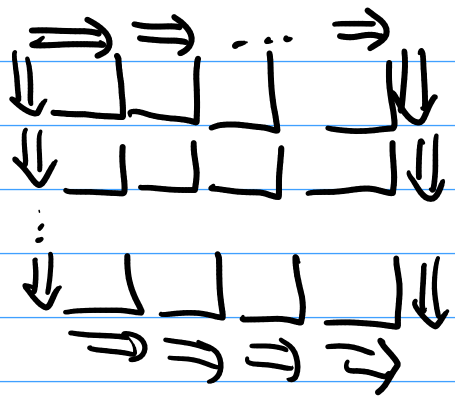
1) $\to \subseteq \Rightarrow$
2) $\Rightarrow \subseteq \twoheadrightarrow$
3) $\Rightarrow$ TIENE LA PROP. DEL DIAMANTE "$\Diamond(\Rightarrow)$".

si $a \Rightarrow b$, $a \Rightarrow c$

$\exists d.\ b \Rightarrow d,\ c \Rightarrow d.$

$$\begin{tikzcd} & a \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ b \arrow[dr, Rightarrow, dotted] & & c \arrow[dl, Rightarrow, dotted] \\ & d & \end{tikzcd}$$





Def. "$\Rightarrow$":

$$\frac{}{x \Rightarrow x} \qquad \frac{t \Rightarrow t'}{\lambda x.t \Rightarrow \lambda x.t'} \qquad \frac{t \Rightarrow t' \quad s \Rightarrow s'}{ts \Rightarrow t's'}$$

$$\frac{t \Rightarrow t' \qquad s \Rightarrow s'}{(\lambda x.t)s \Rightarrow t'\{x := s'\}}$$

$t ::= x \mid \lambda x.t \mid tt$

Programando en C-λ.

1) Booleanos.

$T := \lambda x.\lambda y.x$ $\quad = $ "K"

$F := \lambda x.\lambda y.y$ $\quad$ KI

$IF := \lambda b.\lambda x.\lambda y.(b\,x)y$ $\quad := \boxed{\lambda x.x}$

$=_\eta \lambda b.\lambda x.bx =_\eta \lambda b.b$

VERIFICA:

$IF\ T\ t\ s \twoheadrightarrow t$

$IF\ F\ t\ s \twoheadrightarrow s$

$\underbrace{IF\ T}\ t\ s \rightarrow T\ t\ s \rightarrow (\lambda y.t)s \rightarrow t$

2) Pares:

$\langle t,s\rangle := \lambda x.x\,t\,s$

$\pi_1 := \lambda p.\,pT$

$\pi_2 := \lambda p.\,pF$

VERIFICA:

$\pi_1 \langle t,s\rangle \twoheadrightarrow t$

$\pi_2 \langle t,s\rangle \twoheadrightarrow s$

$\pi_1 \langle t,s\rangle \rightarrow \langle t,s\rangle T \rightarrow T\,t\,s \rightarrow t$

3) Números:

$0 := \lambda x.\lambda y.x$

$S := \lambda n.\lambda x.\lambda y.\,y\,n$

$iszero := \lambda n.\,n\,T(\lambda z.F)$

$pred := \lambda n.\,n\,\Omega(\lambda x.x)$

VERIFICA:

$iszero\ 0 \twoheadrightarrow T$

$iszero(S\,n) \twoheadrightarrow F$

$pred\,(S\,n) \twoheadrightarrow n$

$S\,0 = \lambda x.\lambda y.y\,0$

$S(S\,0) = \lambda x.\lambda y.y(\lambda x.\lambda y.y\,0)$

4) <u>Recursión</u>:

"BASE": $Z = (\lambda x. f(xx))(\lambda x. f(xx))$

$\to fZ \to f(fZ) \to f(f(fZ))$

$\to \dots$

$$Y := \lambda f. \left((\lambda x. f(xx)) (\lambda x. f(xx)) \right)$$

<u>PROPIEDAD</u>: $Yt \twoheadrightarrow t(Yt) \twoheadrightarrow t(t(Yt))$

$$+ := Y(\lambda f. \underbrace{\lambda n. \lambda m. \ \mathrm{IF}\ (\mathrm{isZero}\ n)\ m\ (S(f\ (\mathrm{pred}\ n)\ m))}))$$

<u>VERIFICA</u>:

$+\ 0\ m \twoheadrightarrow m$

$+ (S n)\ m \twoheadrightarrow S(+\ n\ m)$